

Anisotropic elastic theory of preloaded granular media

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Abstract

A macroscopic elastic description of stresses in static, preloaded granular media is derived systematically from the microscopic elasticity of individual inter-grain contacts. The assumed preloaded state and friction at contacts ensure that the network of inter-grain contacts is not altered by small perturbations. The texture of this network, set by the preparation of the system, is encoded in second and fourth order fabric tensors. A small perturbation generates both normal and tangential inter-grain forces, the latter causing grains to reorient. This reorientation response and the incremental stress are expressed in terms of the macroscopic strain.

The transmission of stress in granular media has a rich phenomenology [1, 2, 3], as illustrated by the emblematic sand pile problem. In a conical pile obtained by pouring grains from a point source (hopper outlet), the pressure profile at the base of the pile is not proportional to the height of the pile, as it would if the weight of each grain were transmitted strictly vertically; nor does the pressure vary monotonically from edge to center, as predicted by traditional isotropic elastic or elasto-plastic approaches. Rather, the pressure profile develops a local minimum (often termed ‘pressure dip’) at the center of the pile base, below the apex of the pile [4, 5, 6, 7]. By contrast, if a pile of the same shape is prepared layer by layer, *e.g.*, by sprinkling from an extended source, the pressure profile *does* acquire a maximum at the center of the base (‘pressure hump’) [4, 5, 6, 7]. These and similar experiments indicate that the local structure of the pile (often called ‘texture’), which governs stress transmission, depends

sensitively on the preparation of the system. In two dimensional packings of monodisperse disks [7], for example, the observed distribution of inter-grain contact orientations reflects the local grain ordering, with a clear six-fold modulation. However, as expected from simple symmetry arguments (see Fig. 1 for an illustration), the distribution displays a stronger degree of anisotropy (including two- and four-fold modulations) in a usual pile, built with a point source, than in a pile grown by sprinkling grains. Further experiments [7, 8], as well as simulations [9], confirm that the anisotropy of the *microscopic* contact distribution is correlated with the presence of a *macroscopic* pressure dip.

As it proved difficult to fit the wealth of observed stress profiles [4, 5, 6, 7, 10, 11, 8] with usual elastic theories, in the past decade physicists introduced highly idealized discrete models of static granular media [4, 12, 13, 14] based on probabilistic rules of force transmission between neighbouring grains. Although these models reproduce experimental features, they leave stresses underspecified (and consequently rely on a somewhat *ad hoc* constraint among stress components), as a result of an incomplete treatment of inter-grain forces. Furthermore, careful investigations of force transmission models [15, 16, 17] seem to indicate that disorder in the granular packing might cause it to behave again as an elastic medium at large scales. In this spirit, Ref. [18] reviews the various possible outcomes of an (orthotropic) [19] anisotropic elastic theory [20], without however addressing the origin of the anisotropy in the context of granular media.

In a continuum, any volume element is compactly embedded in the surrounding medium. By contrast, in more fragile granular media, stress is transmitted through isolated inter-grain contacts, and under a perturbation grains slightly reorient with respect to their surroundings. This occurs even in the case of a simple compressive strain: consider for example a compression along the x -axis and focus on the contact of a given grain with a neighbouring grain. The torque that this contact imparts to the grain vanishes only if the contact is parallel or perpendicular to the x -axis. Thus, generically, a non-vanishing total torque is applied on a grain by contacting grains, and consequently it reorients so as to restore the torque to zero in equilibrium. A continuum theory derived from inter-grain forces must include effects of grain reorientation. In this vein, a generalization of usual elasticity that accommodates the ‘micro-rotation’ of points in addition to the ‘macro-rotation’ of the medium was formulated by the Cosserat brothers [21], and subsequently generalized and applied to ‘complex’ continuum media [22, 23].

In the present work, our starting point is the well-established elastic theory of inter-grain contacts [24, 25]; on this basis, we construct a specific macroscopic elastic theory [26] that encodes the texture of the granular network. We consider a preloaded granular medium [27] and small applied incremental stresses, so that the corresponding elastic response be linear. While the unperturbed inter-grain forces reflect the rolling and sliding [28] that occur during the preparation of the system, as well as the external preloading, if any, we expect the response of the medium to small incremental stresses to be sensitive only to properties of the (unperturbed) contact network. Below, we derive general expressions for the (anisotropic) elastic linear response and the associated grain reorientation

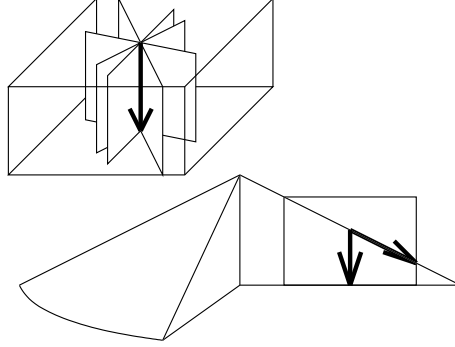


Figure 1: In a pile constructed layer by layer (top), the local texture of the packing has only one preferred (vertical) direction and is therefore statistically symmetric about any vertical plane (and hence invariant under rotation about a vertical axis). By contrast, in a usual pile grown with a point source (bottom), the presence of another special direction (the downhill direction) lowers the symmetry, and the local texture is statistically symmetric with respect to a single vertical plane, that which contains the downhill direction.

field.

The contacting region between two preloaded spherical grains A and B (Fig. (a)) is a disk with diameter $a \approx [3FD(1 - \nu_g^2)/8E]^{1/3}$, where F is the inter-grain compressive force, D the grain diameter (or, equivalently, the centre-to-centre distance), E the Young modulus and ν_g the Poisson ratio of the grain constitutive material [24]. Because of the high friction established by the preloading, essentially no slip occurs at the contact upon a perturbation of the grains [29]. Consequently, the linear elastic response of such a contact is characterized by stiffness constants k_{\parallel} for compression or traction along the centre-to-centre vector \vec{D} (henceforth called *contact vector*) (Fig. (b)) and k_{\perp} for transverse forces

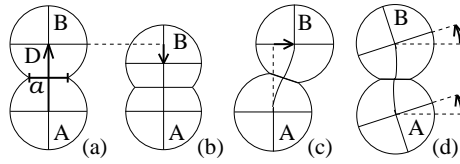


Figure 2: Schematic illustration of the preloading and deformation of two contacting grains. (a) The diameter a of the contact region between two preloaded grains is non-vanishing (over-emphasized on the figure). A restoring force about this compressed state is generated by (b) further compression (or traction), (c) shear, or (d) reorientation. If $a \ll D$, (c) and (d) are equivalent for small perturbations. Correspondingly, tensors ε and Ω enter Eq. (3) on the same footing.

due to shearing (Fig. (c)) [25]. Transverse forces are also generated by the reorientation of two contacting grains (Fig. (d)). Typically, $a \ll D$, implying $k_{\parallel}, k_{\perp} \sim Ea$, and the ratio k_{\perp}/k_{\parallel} is fixed by ν_g ; for example, for a purely compressively preloaded contact $k_{\perp}/k_{\parallel} = (2 - 2\nu_g)/(2 - \nu_g)$ [25]. In addition to compressive and shearing forces, torques may be transmitted through twisted inter-grain contacts, with a corresponding stiffness $k_t \sim Ea^3$ relating the twist angle to the torque. As long as $a \ll D$, such twisting torques contribute negligibly to the behaviour of the medium and we neglect them in the present work. (In the case of non-spherical (e.g., faceted) grains, in which twisting modes may be more relevant, one can keep track of them by defining a torque flux tensor, often termed ‘couple stress’ in Cosserat theories [21, 22], and proceed as with the stress tensor defined below.) Combining these various elements, we write the force exerted by grain B on grain A as

$$f_i^{\text{AB}} = k_{ij}^{\text{AB}} (u_j^{\text{B}} - u_j^{\text{A}}) - k_{ij}^{\text{AB}} \frac{\Omega_{jk}^{\text{A}} + \Omega_{jk}^{\text{B}}}{2} D_k^{\text{AB}}, \quad (1)$$

where

$$k_{ij}^{\text{AB}} = k_{\parallel} \frac{D_i^{\text{AB}} D_j^{\text{AB}}}{D^2} + k_{\perp} \left(\delta_{ij} - \frac{D_i^{\text{AB}} D_j^{\text{AB}}}{D^2} \right) \quad (2)$$

and the sum over repeated indices is understood. The tensor Ω^{G} is antisymmetric and is defined such that the rotation matrix $[\exp(\Omega^{\text{G}})]_{ij} \approx \delta_{ij} + \Omega_{ij}^{\text{G}}$ describes the *reorientation* of grain G. We note that the mean reorientation, $(\Omega^{\text{A}} + \Omega^{\text{B}})/2$, results in a sheared contact (Fig. (d)) and thus contributes to the force (Eq. (1)), while the difference $\Omega^{\text{A}} - \Omega^{\text{B}}$ corresponds to grain rolling, which does not affect the force between spherical grains [30].

In order to build a continuum description of a preloaded granular medium, we identify the displacements \vec{u}^{G} of the centers of the grains (labelled by G) with a smoothly varying, macroscopic displacement field $\vec{u}(\vec{r})$. Similarly, we identify the grain reorientation Ω^{G} with a smooth reorientation field $\Omega(\vec{r})$, to be determined. In doing so, we neglect the non-affine component of the grain displacements. This approximation is exact in the limit of a simple crystalline granular network, in which all grains are equivalent, but we expect it to be good even away from this limit. If the packing disorder becomes too important, the medium can still be described by an elastic theory, although its effective stiffness constants may depart significantly from those obtained below [31]. Grain displacements can now be expressed in terms of the strain tensor, as $u_i^{\text{B}} - u_i^{\text{A}} = \varepsilon_{ij} D_j^{\text{AB}}$, where $\varepsilon_{jk} = (\partial_j u_k + \partial_k u_j)/2$ as usual, and the force transmitted through a contact whose contact vector lies along the direction α reads

$$f_i(\alpha) = k_{ij}(\alpha) (\varepsilon_{jk} - \Omega_{jk}) D_k(\alpha), \quad (3)$$

where $k_{ij}(\alpha)$ and $D_k(\alpha)$ refer to a contact normal to the direction α . We emphasize that, as it appears explicitly in Eq. (3), the grain reorientation Ω affects inter-grain forces; it does *not* represent the antisymmetric part of the displacement gradient (which corresponds to solid body rotation and does not generate stress).

For a description in terms of stresses, it is convenient to define the average number $\mu(\alpha)$ $d\alpha$ of contact vectors that lie within a solid angle $d\alpha$ about direction α , per unit volume, and to introduce local *fabric tensors* [2] Q and P whose slow spatial variations reflect those of D , k_{\parallel} , k_{\perp} , and μ , as

$$Q_{kj} = \int k_{\perp} D_k D_j \mu d\alpha \quad (4)$$

and

$$P_{ijkl} = \int (k_{\parallel} - k_{\perp}) \frac{D_i D_j D_k D_l}{D^2} \mu d\alpha. \quad (5)$$

In equilibrium, the total torque $\sum_{G'} \vec{f}^{GG'} \times \vec{D}^{GG'}$ imparted to grain G by its neighbours (labelled by G') vanishes. Locally averaged in space, this condition becomes

$$\int \vec{f}(\alpha) \times \vec{D}(\alpha) \mu(\alpha) d\alpha = 0 \quad (6)$$

or, equivalently,

$$\Omega Q + Q \Omega = \varepsilon Q - Q \varepsilon. \quad (7)$$

This identity will be useful in the derivation of the stress in terms of the strain only, and confirms that grains reorient unless their contacts lie along a principal axis of the strain (on average), *i.e.*, unless ε and Q commute. In order to compute the stress, we note that $D_j(\alpha) \mu(\alpha) d\alpha$ is the number of contact vectors per unit surface area, oriented within $d\alpha$ about direction α , that intersect a surface normal to direction j . Since $f_i(\alpha)$ is the i -th component of the force transmitted along such contact vectors, the stress tensor can be written as

$$\sigma_{ij} = \int f_i(\alpha) D_j(\alpha) \mu(\alpha) d\alpha. \quad (8)$$

While this expression is not symmetric in general, it is symmetrized as expected by the above vanishing-torque condition (Eq. (7)). Combining Eqs. (3–8), we obtain

$$\Omega = \int_0^\infty e^{-sQ} (\varepsilon Q - Q \varepsilon) e^{-sQ} ds \quad \text{or} \quad \Omega_{ij} = \frac{q_j - q_i}{q_j + q_i} \varepsilon_{ij}, \quad (9)$$

and

$$\sigma = \int_0^\infty e^{-sQ} 2Q \varepsilon Q e^{-sQ} ds + P : \varepsilon \quad \text{or} \quad \sigma_{ij} = 2 \frac{q_i q_j}{q_i + q_j} \varepsilon_{ij} + P_{ijkl} \varepsilon_{kl}. \quad (10)$$

The right-hand expressions refer to a basis in which Q is diagonal, and q_i is the i -th eigenvalue of Q ; indices i and j are *not* summed over. As expected, the reorientation, Ω , depends on k_{\perp} (through Q) but not on k_{\parallel} . Equation (9) is qualitatively corroborated by the experimental observation [32] that Ω is largest in regions of large deformation.

Equations (9,10) constitute our central result and describe the response of a preloaded granular medium to an incremental strain. If the medium is statistically isotropic ($\mu = \rho/4\pi$ constant), we recover standard isotropic elasticity, with Lamé coefficients $\lambda_L = \rho D^2(k_{\parallel} - k_{\perp})/15$ and $\mu_L = \rho D^2(k_{\parallel}/15 + k_{\perp}/10)$, in agreement with the results of Chang and Gao [23, 26]. In particular, if the Poisson ratio ν_g of the grain constitutive material is positive, that of the preloaded granular medium, $\nu = \nu_g/(10 - 6\nu_g)$, lies between 0 and 1/14. (We recall, however, that in the isotropic limit our coarse-grained theory is only approximate). In an anisotropic medium, the fabric tensors Q and P encode the texture of the medium through the dependence of D , k_{\parallel} , k_{\perp} , and μ on α . Indeed, if μ varies with α or if the preloading stress is anisotropic, stiffness constants may also vary with α as contacts are then more or less compressed depending on their orientations and may be sheared. The elastic response described by Eq. (10) involves only the first few multipolar components of the quantities $D^2 k_{\perp} \mu$ (through Q) and $D^2(k_{\parallel} - k_{\perp})\mu$ (through P). Monopoles contribute to the isotropic part of the response. Dipoles are absent due to the fore-aft symmetry of the contact distribution. Quadrupoles in Q and P and octupoles in P contribute to the anisotropic elastic response. Higher order multipoles play no role in the present linear theory. As to reorientation, given by Eq. (9), the response is affected solely by the quadrupolar component of $D^2 k_{\perp} \mu$.

Before concluding, we mention that it is possible to construct an iterative scheme for calculating stresses in a sand pile. It relies on the assumption that the mechanical noise due to avalanches of grains at the free surface, and other factors such as temperature fluctuations, do not cause significant rearrangements of the contact network [3]. A rigorous implementation of an iterative scheme still requires knowledge of the complicated mechanisms involved in the growth of the pile, which include both grain avalanching [33] and the non-trivial frictional mechanics putting grains to rest [28, 34]. Indeed, these determine both the angular contact distribution and the tangential compressive stresses at the free surface.

As a first attempt at a quantitative understanding, one may focus on the simpler problem of calculating the macroscopic response to an applied point force [19, 20, 14, 9, 35, 18] for different possible textures of the granular medium. Using Eqs. (9, 10), we find that stress transmission is very sensitive to the texture as summarized by tensors Q and P , and comes in qualitatively diverse forms. Specifically, depending on the degree of anisotropy, the incremental stress profile may be single- or double-peaked [35], in agreement with Refs. [9, 18]; furthermore, the width of the peaks depends also on the degree of anisotropy and, in the single-peak case, may depart from its isotropic counterpart [35].

In sum, we have derived a macroscopic formulation of stresses in granular media. This formulation differs from earlier ones in that it incorporates the possibly anisotropic texture of the granular network as well as the reorientation of grains induced by macroscopic deformations. Moreover, the central objects in our formulation, the fabric tensors, are defined in terms of the microscopic parameters that characterize inter-grain contacts.

One can wonder whether this description might bear some signature of a

salient experimental fact: the existence of highly stressed regions known as ‘force chains’ [4, 10, 8], arranged in a percolating, filamentous network that appears to convey a large fraction of the stress through the medium. Analogously [36, 37] to force chains, stresses in a crumpled elastic sheet are confined mostly within narrow regions, the folds, if the system is allowed to bend in the third dimension (of the embedding space) [36, 37]. In our case, grain reorientation may be viewed as an additional degree of freedom which the system uses to relieve stress. It would be interesting to investigate whether this reorientational freedom of grains may favor, in a heterogeneous system, stress condensation reminiscent of force chains.

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References

- [1] For recent reviews, see R.M. Nedderman, *Statics and Kinematics of Granular Materials* (Cambridge Univ. Press, Cambridge, 1992); S.B. Savage, in *Physics of Dry Granular Media*, H.J. Herrmann, J.-P. Hovi, and S. Luding, Eds. (Kluwer Academic Publishers, Dordrecht, 1998); J.-P. Bouchaud, Les Houches LXXVII lecture notes, 2002 (cond-mat/0211196).
- [2] J.D. Goddard, in *Physics of Dry Granular Media*, H.J. Herrmann, J.-P. Hovi, and S. Luding, Eds. (Kluwer Academic Publishers, Dordrecht, 1998);
- [3] P.-G. de Gennes, *Physica A* **261**, 267 (1998).
- [4] C.-H. Liu, S.R. Nagel, D.A. Scheeter, S.N. Coppersmith, S. Majumdar, O. Narayan, and T.A. Witten, *Science* **269**, 513 (1995).
- [5] R. Brockbank, J.M. Huntley, and R.C. Ball, *J. Phys. (France) II* **7**, 1521 (1997).
- [6] L. Vanel, D.W. Howell, D. Clark, R.P. Behringer, and E. Clément, *Phys. Rev. E* **60**, R5040 (1999).
- [7] J. Geng, E. Longhi, R.P. Behringer, and D.W. Howell, *Phys. Rev. E* **64**, 060301 (2001).
- [8] J. Geng *et al.*, *Phys. Rev. Lett.* **87**, 035506 (2001).

- [9] C. Goldenberg and I. Goldhirsch, *Phys. Rev. Lett.* **89**, 084302 (2002); I. Goldhirsch and C. Goldenberg, *Preprint* (cond-mat/0201081).
- [10] E. Clément, G. Reydellet, L. Vanel, D.W. Howell, J. Geng, and R.P. Behringer, in *XIII Internat. Congr. on Rheology, Cambridge, UK*, vol. **2** (British Soc. of Rheology, Glasgow, 2000).
- [11] G. Reydellet and E. Clément, *Phys. Rev. Lett.* **86**, 3308 (2001).
- [12] S.N. Coppersmith, C.-H. Liu, S. Majumdar, O. Narayan, and T.A. Witten, *Phys. Rev. E* **53**, 4673 (1996).
- [13] J.-P. Bouchaud, M.E. Cates, and P. Claudin, *J. Phys. I* **5**, 639 (1995); J.P. Wittmer, M.E. Cates, P. Claudin, and J.-P. Bouchaud, *Nature* **382**, 336 (1996); J.P. Wittmer, M.E. Cates, and P. Claudin, *J. Phys. I* **7**, 39 (1997).
- [14] P. Claudin, J.-P. Bouchaud, M.E. Cates, and J.P. Wittmer, *Phys. Rev. E* **57**, 4441 (1998).
- [15] J.-P. Bouchaud, P. Claudin, D. Levine, and M. Otto, *Eur. Phys. J. E* **4**, 451 (2001).
- [16] J.E.S. Socolar, D.G. Schaeffer, and P. Claudin, *Eur. Phys. J. E* **7**, 353 (2002).
- [17] J.E.S. Socolar, *Disc. and Cont. Dyn. Sys. B*, in press (2003) (cond-mat/0212162).
- [18] M. Otto, J.-P. Bouchaud, P. Claudin, and J.E.S. Socolar, *Phys. Rev. E* **67**, 031302 (2003).
- [19] A.E. Green and G.I. Taylor, *Proc. Roy. Soc. Lond. A* **173**, 162 (1939).
- [20] A.E. Green and W. Zerna, *Theoretical Elasticity* (Oxford Univ. Press, London, 1954).
- [21] E. Cosserat and F. Cosserat, *Théorie des Corps Déformables* (Hermann et Fils, Paris, 1909).
- [22] R. Lakes, in *Continuum models for materials with micro-structure*, H. Mühlhaus Ed. (Wiley, New York, 1995).
- [23] C.S. Chang and J. Gao, *Int. J. Solids Structures* **32**, 2279 (1995).
- [24] H. Hertz, *J. reine und angewandte Mathematik* **92**, 156 (1882).
- [25] K.L. Johnson, *Contact Mechanics* (Cambridge Univ. Press, Cambridge, 1985).

- [26] According to the classification of Chang and Gao [23], our theory describes a ‘micro-polar continuum’; however, importantly, in our case the continuum is anisotropic. Because our grains are spherical and we neglect torques transmitted by inter-grain contacts, the medium further reduces to an anisotropic ‘quasi-micro-polar continuum.’ When our description is applied to a statistically isotropic medium, with vanishing grain reorientation according to Eq. (9), it coincides with the first order of Chang and Gao’s isotropic ‘non-polar theory’ [23]. In similar approaches but that now include anisotropy, Emeriault and Chang (*J. Eng. Mech. ASCE* **123**, 1289 (1997)) treat non-linear responses, and Emeriault and Claquin (*Granular Matter* **2**, 201 (2000)) focus on (linear) viscoelastic responses. In neither case, however, is grain reorientation involved in relating microscopic displacements to macroscopic strains.
- [27] In experiments without external preloading, the medium is still preloaded by its own weight, albeit in a non-uniform fashion. Since our description is local (*i.e.*, fabric tensors may vary in space), this non-uniformity does not present a difficulty.
- [28] F. Radjai, M. Jean, J.-J. Moreau, and S. Roux, *Phys. Rev. Lett.* **77**, 274 (1996).
- [29] A slight shearing of a preloaded contact causes slip only at its periphery [25]. Since the contact area is barely affected, the stiffness constants k_{\parallel} and k_{\perp} do not vary to first order.
- [30] Furthermore, loops with an odd number of grains frustrate grain rolling, and presumably suppress variations of the reorientation. On loops and force balance, see R.C. Ball and R. Blumenfeld, *Phys. Rev. Lett.* **88**, 115505 (2002).
- [31] Recent simulations of particles interacting *via* a central pair potential, with periodic boundary conditions in two dimensions (A. Tanguy, J.P. Wittmer, F. Leonforte, and J.-L. Barrat, *Phys. Rev. B* **66**, 174205 (2002); J.P. Wittmer, A. Tanguy, J.-L. Barrat, and L. Lewis, *Europhys. Lett.* **57**, 423 (2002)), indicate that non-affine displacements may affect the elastic properties of a disordered medium up to scales of about 30 interparticle distances. Extensions of this result to three dimensions or to include tangential forces are not yet available.
- [32] F. Calvetti, G. Combe, and J. Lanier, *Mechanics of Cohesive-Frictional Materials* **2**, 121 (1997).
- [33] J. Rajchenbach, *Advances in Physics* **49**, 229 (2000); *Phys. Rev. Lett.* **88**, 014301 and **89**, 074301 (2002).
- [34] T.C. Halsey and D. Ertaş, *Phys. Rev. Lett.* **83**, 5007 (1999).
- [35] R. A. da Silveira, G. Vidalenc, and C. Gay, *Preprint* (cond-mat/0208214).

- [36] T.A. Witten, lecture series, ESPCI, Paris (1999).
- [37] B.A. DiDonna, T.A. Witten, S.C. Venkataramani, and E.M. Kramer, *Phys. Rev. E.* **65**, 016603 (2002).